

THE BOUNDARY BEHAVIOR AND UNIQUENESS OF SOLUTIONS OF THE HEAT EQUATION

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1. Introduction. In this paper we consider some questions on the boundary behavior and uniqueness of solutions of the one-dimensional heat equation in the infinite strip $0 < t < c$ and in the half plane $0 < t < \infty$. For convenience we adopt the following notation.

Suppose that $u = u(x, t)$ is defined over some domain δ in the xt -plane. We say that $u \in H$ in δ if u has continuous second partial derivatives and if $u_t = u_{xx}$ at each point of δ . We say that $u \in H^+$ in δ if, in addition, $u \geq 0$ in δ and that $u \in H^\Delta$ in δ if $u = u_1 - u_2$ where $u_1, u_2 \in H^+$ in δ . Clearly $H^+ \subset H^\Delta$.

By a well known theorem due to Widder [19] $u \in H^\Delta$ in $0 < t < c$ if and only if u has the representation

$$(1.1) \quad u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y)$$

in $0 < t < c$, where $\alpha = \alpha(x)$ has bounded variation over each finite interval, where the integral is absolutely convergent in this strip, and where $k(x, t)$ is the Poisson kernel

$$k(x, t) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}.$$

(See also [7].) If we assume, as we clearly may, that α is normalized, i.e.

$$(1.2) \quad \alpha(x) = \frac{\alpha(x + 0) + \alpha(x - 0)}{2}$$

for all x , then

$$(1.3) \quad \alpha(x_2) - \alpha(x_1) = \lim_{t \rightarrow 0+} \int_{x_1}^{x_2} u(y, t) dy$$

for all x_1, x_2 . (See [3] or [21]. This is also a simple consequence of Theorem 1.) Hence with each $u \in H^\Delta$ in $0 < t < c$ we can associate an α , unique except for an additive constant, satisfying (1.1), (1.2), and (1.3). If we think of $u(x, t)$ as the temperature of an infinite insulated rod, extended along the x -axis, at the point x of the rod and at time t , then $\alpha(x)$ represents the heat distribution in the rod at time $t = 0$.

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Solutions of the heat equation and harmonic functions have many similar properties and in this paper we obtain the analogues, for functions in H^A over $0 < t < c$, of several well known theorems concerning functions which are harmonic in a half plane or, alternatively, which are harmonic in the unit circle. In §2 we consider various forms of the Fatou theorem and obtain information on the behavior of $u(x, t)$ as (x, t) approaches $(x_0, 0)$ from information on the behavior of $\alpha(x)$ near x_0 . A simple example shows that, in general, it is not possible to invert the results of §2 and obtain information on the behavior of $\alpha(x)$ near x_0 from the behavior of $u(x, t)$ near $(x_0, 0)$. However, in §3 and §4, we restrict our attention to the important subclass of functions in H^+ and we obtain the "corrected converses" or Tauberian theorems corresponding to the Abelian theorems of §2. (For functions in H^A and in H^+ over $0 < t < \infty$ we establish analogous relations between the behavior of $u(x, t)$ as $t \rightarrow \infty$ and the behavior of $\alpha(x)$ for large x .) In §5 we list some uniqueness theorems and in §6 we give an analogue for an inequality due to Fejér and F. Riesz.

2. Fatou theorem. For each $a > 0$ we let $S(a, x_0)$ and $P(a, x_0)$ denote, respectively, the following sector and parabolic domains:

$$S(a, x_0) = \{(x, t): |x - x_0| < at, t > 0\},$$

$$P(a, x_0) = \{(x, t): |x - x_0| < at^{1/2}, t > 0\}.$$

From the identity

$$\int_{-\infty}^{\infty} \left| \frac{x}{2t^{1/2}} \right|^n |k(x, t)| dx = \frac{1}{\pi^{1/2}} \Gamma\left(\frac{n+1}{2}\right), \quad n > -1,$$

it is not difficult to show that, for each $a > 0$, there exists a constant $C_1 = C_1(a) < \infty$ such that

$$(2.1) \quad \int_{-\infty}^{\infty} |yk_v(x - y, t)| dy \leq C_1, \quad \int_{-\infty}^{\infty} |yk_{vv}(x - y, t)| dy \leq \frac{C_1}{t^{1/2}}$$

for all $(x, t) \in P(a, 0)$.

For functions in H^A we have the following "localization" theorem.

LEMMA 1. Suppose that $u \in H^A$ in $0 < t < c$ and that $d > 0$. Then, for each x_0 ,

$$(2.2) \quad \int_{|y-x_0|>d} k(x - y, t) d\alpha(y) = o(1), \quad \int_{|y-x_0|>d} k_z(x - y, t) d\alpha(y) = o\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $t > 0$, and

$$(2.3) \quad \int_{|y-x_0|<d} k(x - y, t) d\alpha(y) = O\left(\frac{1}{t^{1/2}}\right),$$

$$\int_{|y-x_0|<d} k_z(x - y, t) d\alpha(y) = O\left(\frac{1}{t}\right)$$

as $t \rightarrow \infty$, uniformly in x .

Proof. For the first part of (2.2) fix $0 < b < c$. The inequalities $|x - x_0| < d/2$, $|y - x_0| > d$, $0 < t < b/8$ imply that

$$k(x - y, t) \leq \frac{1}{(4\pi t)^{1/2}} e^{-d^2/32t} e^{-(x_0 - y)^2/4b} = C_2 k(d, 8t) k(x_0 - y, b)$$

where $C_2 = (32\pi b)^{1/2}$. Hence we obtain

$$\left| \int_{|y - x_0| > d} k(x - y, t) d\alpha(y) \right| \leq C_2 k(d, 8t) \int_{-\infty}^{\infty} k(x_0 - y, b) |d\alpha(y)|$$

for $|x - x_0| < d/2$, $0 < t < b/8$, and, since $k(d, 8t) \rightarrow 0$ as $t \rightarrow 0$, the desired conclusion follows directly. A similar argument yields the second part of (2.2), and (2.3) is an immediate consequence of the inequalities,

$$(2.4) \quad k(x, t) \leq \frac{1}{(4\pi t)^{1/2}}, \quad |k_x(x, t)| \leq \frac{1}{(4\pi)^{1/2} t}$$

for all (x, t) in $0 < t < \infty$.

Functions in H^A satisfy the following order condition near the line $t = 0$.

LEMMA 2. Suppose that $u \in H^A$ in $0 < t < c$. If α is continuous at x_0 , then

$$(2.5) \quad u(x, t) = o\left(\frac{1}{t^{1/2}}\right), \quad u_x(x, t) = o\left(\frac{1}{t}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $t > 0$. If α has a jump of λ at x_0 , then, for each $a > 0$,

$$(2.6) \quad u(x, t) \sim \frac{\lambda}{(4\pi t)^{1/2}}, \quad u_x(x, t) = o\left(\frac{1}{t}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in S(a, x_0)$.

Proof. If α is continuous at x_0 , we can find a $d > 0$ such that

$$\int_{|y - x_0| \leq d} |d\alpha(y)| < \epsilon.$$

Next we can differentiate both sides of (1.1) [19] and, with (2.2) and (2.4), we conclude that

$$|u(x, t)| \leq \frac{\epsilon}{(4\pi t)^{1/2}} + o(1), \quad |u_x(x, t)| \leq \frac{\epsilon}{(4\pi)^{1/2} t} + o\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $t > 0$. Hence (2.5) follows. If α has a jump of λ at x_0 , we write $\alpha = \alpha_c + \alpha_j$, where α_c is continuous at x_0 and where α_j is constant except for a jump of λ at x_0 . Then

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha_e(y) + \lambda k(x - x_0, t),$$

and (2.6) follows from (2.5).

We consider next the following two versions of the Fatou theorem for functions in H^Δ . The $(x, t) \rightarrow (x_0, 0)$ parts of each of these results are known [15] but proofs are included for the sake of completeness. We let $D\alpha(x_0)$ denote the symmetric derivative of α at x_0 , i.e.

$$D\alpha(x_0) = \lim_{h \rightarrow 0} \frac{\alpha(x_0 + h) - \alpha(x_0 - h)}{2h}.$$

THEOREM 1. Suppose that $u \in H^\Delta$ in $0 < t < c$. If $D\alpha(x_0) = A$, then, for each $a > 0$,

$$(2.7) \quad u(x, t) \rightarrow A$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in S(a, x_0)$. Suppose that $u \in H^\Delta$ in $0 < t < \infty$. If

$$\frac{\alpha(x_0 + x) - \alpha(x_0 - x)}{2x} \rightarrow A$$

as $x \rightarrow \infty$, then $u(x_0, t) \rightarrow A$ as $t \rightarrow \infty$.

THEOREM 2. Suppose that $u \in H^\Delta$ in $0 < t < c$. If $\alpha'(x_0) = A$, then, for each $a > 0$,

$$(2.8) \quad u(x, t) \rightarrow A, \quad u_x(x, t) = o\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in P(a, x_0)$. Suppose that $u \in H^\Delta$ in $0 < t < \infty$. If

$$\frac{\alpha(x_0 + x) - \alpha(x_0)}{x} \rightarrow A$$

as $|x| \rightarrow \infty$, then, for each $a > 0$, (2.8) holds as $t \rightarrow \infty$, $(x, t) \in P(a, x_0)$.

Proof of Theorem 1. If we replace $\alpha(x)$ by $\alpha(x) - \alpha(x_0)$ and x by $x - x_0$, we see we can assume that $\alpha(x_0) = 0$ and that $x_0 = 0$. The hypotheses for the first part of the theorem are then that

$$(2.9) \quad \frac{\alpha(x) - \alpha(-x)}{2x} \rightarrow A$$

as $x \rightarrow 0+$. If we let

$$(2.10) \quad z = y^2, \quad s = \frac{1}{4t}, \quad \beta(z) = \alpha(y) - \alpha(-y),$$

we can write

$$(2.11) \quad u(0, t) = \int_{-\infty}^{\infty} k(y, t) d\alpha(y) = \left(\frac{s}{\pi}\right)^{1/2} \int_0^{\infty} e^{-sz} d\beta(z).$$

Because of (2.2), we can assume that $\alpha(x)$ is constant for large x and hence that the Laplace integral in (2.11) converges for $0 < s < \infty$. From (2.9) we see that

$$(2.12) \quad \frac{\beta(z)}{z^{1/2}} \rightarrow 2A$$

as $z \rightarrow 0+$, and applying a well known Abelian theorem for the Laplace transform [20, p. 182] we conclude that

$$(2.13) \quad \left(\frac{s}{\pi}\right)^{1/2} \int_0^{\infty} e^{-sz} d\beta(z) \rightarrow A$$

as $s \rightarrow \infty$. Hence

$$u(0, t) \rightarrow A$$

as $t \rightarrow 0+$, and the convergence in $S(a, 0)$ is an immediate consequence of Lemma 2.

The hypotheses for the second part of Theorem 1 imply that the Laplace integral in (2.11) converges for $0 < s < \infty$ and that (2.12) holds as $z \rightarrow \infty$. Appealing to the same Abelian theorem we obtain (2.13) as $s \rightarrow 0+$ and this completes the proof.

Proof of Theorem 2. As in the proof for Theorem 1 we can assume that $\alpha(x_0) = 0$, $x_0 = 0$. Furthermore, since

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\{\alpha(y) - Ay\} + A,$$

we need only consider the case where $A = 0$.

The hypotheses for the first part of Theorem 2 imply we can find a $d > 0$ such that

$$(2.14) \quad |\alpha(x)| \leq \epsilon |x|, \quad \epsilon > 0,$$

for $|x| \leq d$. Then (2.2) and an integration by parts yield

$$(2.15) \quad \begin{aligned} u(x, t) &= - \int_{|y| \leq d} k_y(x - y, t) \alpha(y) dy + o(1), \\ u_x(x, t) &= + \int_{|y| \leq d} k_{xy}(x - y, t) \alpha(y) dy + o\left(\frac{1}{t^{1/2}}\right) \end{aligned}$$

as $(x, t) \rightarrow (0, 0)$, $t > 0$. Appealing to (2.14) and (2.1), we conclude that

$$|u(x, t)| \leq \epsilon \int_{-\infty}^{\infty} |yk_v(x - y, t)| dy + o(1) \leq \epsilon C_1 + o(1),$$

$$|u_x(x, t)| \leq \epsilon \int_{-\infty}^{\infty} |yk_{vv}(x - y, t)| dy + o\left(\frac{1}{t^{1/2}}\right) \leq \frac{\epsilon C_1}{t^{1/2}} + o\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (0, 0)$, $(x, t) \in P(a, 0)$. This completes the argument.

The hypotheses for the second part of Theorem 2 imply that (2.14) holds for $|x| \geq d$. From (2.3) and integration by parts, we obtain (2.15) as $t \rightarrow \infty$, uniformly in x , with the integration taken over the range $|y| \geq d$. The proof is then completed as before.

We will require the following results in §§3-4 and in §5, respectively. They can be obtained with trivial modifications of the argument used in the proof of Theorem 2.

COROLLARY 1. *Suppose that $u \in H^{\Delta}$ in $0 < t < c$. If*

$$(2.16) \quad \frac{\alpha(x_0 + x) - \alpha(x_0)}{x} = O(1)$$

as $x \rightarrow 0$, then, for each $a > 0$,

$$(2.17) \quad u(x, t) = O(1), \quad u_x(x, t) = O\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in P(a, x_0)$. Suppose that $u \in H^{\Delta}$ in $0 < t < \infty$. If (2.16) holds as $|x| \rightarrow \infty$, then, for each $a > 0$, (2.17) holds as $t \rightarrow \infty$, $(x, t) \in P(a, x_0)$.

COROLLARY 2. *Suppose that $u \in H^{\Delta}$ in $0 < t < c$. If $\alpha'(x_0) = \infty$ ($-\infty$), then*

$$u(x_0, t) \rightarrow \infty \text{ } (-\infty)$$

as $t \rightarrow 0+$.

3. Converses for the Fatou theorem. We begin with an example which shows that, in general, Theorems 1 and 2 cannot be inverted (cf. [12, p. 246]).

THEOREM 3. *There exists a function $u \in H^{\Delta}$ in $0 < t < \infty$ such that, for each $a > 0$,*

$$(3.1) \quad u(x, t) \rightarrow 0, \quad u_x(x, t) = o\left(\frac{1}{t^{1/2}}\right)$$

as $(x, t) \rightarrow (0, 0)$, $(x, t) \in P(a, 0)$, and as $t \rightarrow \infty$, $(x, t) \in P(a, 0)$, and such that

$$(3.2) \quad \liminf \frac{\alpha(x) - \alpha(-x)}{2x} < \limsup \frac{\alpha(x) - \alpha(-x)}{2x}$$

as $x \rightarrow 0+$ and as $x \rightarrow \infty$.

Proof. Let α be the normalized function defined as follows:

$$\alpha(x) = \begin{cases} 2^n & 2^n < x < 2^n + b_n, \\ 2^{n-1} & x = 2^n, 2^n + b_n, \\ 0 & \text{everywhere else.} \end{cases} \quad n = 0, \pm 1, \pm 2, \dots$$

The b_n are chosen so that $0 < b_n < 2^n$ and so that

$$(3.3) \quad \sum_{n=-\infty}^{\infty} 2^{-n} b_n < \infty.$$

Then (3.2) holds as $x \rightarrow 0+$ and as $x \rightarrow \infty$. With the Law of the Mean we obtain

$$(3.4) \quad u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y) = \sum_{n=-\infty}^{\infty} 2^n b_n k_x(x - y_n, t),$$

where $2^n < y_n < 2^n + b_n < 2^{n+1}$. Fix $a > 0$. It is not difficult to see that

$$\sup_{(x, t) \in P(a, 0)} |k_x(x - 2y, t)| = \frac{1}{4} \sup_{(x, t) \in P(a, 0)} |k_x(x - y, t)| < \infty$$

for each $y \neq 0$. Hence for $(x, t) \in P(a, 0)$ we have

$$(3.5) \quad \begin{aligned} |k_x(x - y_n, t)| &\leq \sup_{(x, t) \in P(a, 0)} |k_x(x - y_n, t)| = 4^{-n} \sup_{(x, t) \in P(a, 0)} |k_x(x - 2^{-n} y_n, t)| \\ &\leq 4^{-n} \sup_{(x, t) \in P(a, 0); 1 < y < 2} |k_x(x - y, t)| = 4^{-n} C_3 \end{aligned}$$

for $n = 0, \pm 1, \pm 2, \dots$, where $C_3 = C_3(a) < \infty$. From (3.3) and (3.5) we conclude that the series in (3.4) is uniformly convergent in $P(a, 0)$. Since

$$k_x(x - y_n, t) \rightarrow 0$$

as $(x, t) \rightarrow (0, 0)$, $t > 0$, and as $t \rightarrow \infty$, uniformly in x , the first part of (3.1) follows directly. The above argument can be modified to show that the series

$$t^{1/2} u_{xx}(x, t) = \sum_{n=-\infty}^{\infty} 2^n b_n t^{1/2} k_{xx}(x - z_n, t), \quad 2^n < z_n < 2^n + b_n$$

is uniformly convergent in $P(a, 0)$, and the second part of (3.1) follows from the fact that

$$t^{1/2} k_{xx}(x - z_n, t) \rightarrow 0$$

as $(x, t) \rightarrow (0, 0)$, $t > 0$, and as $t \rightarrow \infty$, uniformly in x .

Theorem 3 shows that the unrestricted converses for Theorems 1 and 2 are false. However, if we consider the important subclass H^+ , the situation is quite different and we have the following results. (Cf. [6] and [12].)

THEOREM 4. Suppose that $u \in H^+$ in $0 < t < c$. If, for some a ,

$$(3.6) \quad u(x_0 + at, t) \rightarrow A$$

as $t \rightarrow 0+$, then $D\alpha(x_0) = A$. Suppose that $u \in H^+$ in $0 < t < \infty$. If

$$u(x_0, t) \rightarrow A$$

as $t \rightarrow \infty$, then

$$\frac{\alpha(x_0 + x) - \alpha(x_0 - x)}{2x} \rightarrow A$$

as $x \rightarrow \infty$.

THEOREM 5. Suppose that $u \in H^+$ in $0 < t < c$. If, for some a and b ,

$$(3.7) \quad u(x_0 + at^{1/2}, t) \rightarrow A, \quad u(x_0 + bt^{1/2}, t) \rightarrow A, \quad a \neq b,$$

as $t \rightarrow 0+$, then $\alpha'(x_0) = A$. Suppose that $u \in H^+$ in $0 < t < \infty$. If, for some a and b , (3.7) holds as $t \rightarrow \infty$, then

$$(3.8) \quad \frac{\alpha(x_0 + x) - \alpha(x_0)}{x} \rightarrow A$$

as $|x| \rightarrow \infty$.

THEOREM 6. Suppose that $u \in H^+$ in $0 < t < c$. If, for some a and b ,

$$(3.9) \quad u(x_0 + at^{1/2}, t) \rightarrow A, \quad u_x(x_0 + bt^{1/2}, t) = o\left(\frac{1}{t^{1/2}}\right)$$

as $t \rightarrow 0+$, then $\alpha'(x_0) = A$. Suppose that $u \in H^+$ in $0 < t < \infty$. If, for some a and b , (3.9) holds as $t \rightarrow \infty$, then (3.8) holds as $|x| \rightarrow \infty$.

Proof of Theorem 4. Assume that $\alpha(x_0) = 0$, $x_0 = 0$, and consider the first part of Theorem 4. From (3.6) and Lemma 2 we see that

$$u(0, t) \rightarrow A$$

as $t \rightarrow 0+$. Now define z, s, β , as in (2.10). The hypothesis that $u \in H^+$ implies that β is nondecreasing and, with (2.2), we can assume that the Laplace integral in (2.11) is convergent for $0 < s < \infty$. From the above we conclude that (2.13) holds as $s \rightarrow \infty$ and we obtain (2.12), as $z \rightarrow 0+$, from a well known Tauberian theorem for the Laplace transform [20, p. 192]. This completes the proof for the first part of Theorem 4. The argument for the second part is very similar.

The proofs for Theorems 5 and 6 are more complicated and depend upon a number of preliminary results.

We begin with some definitions [8]. We say that $f \in M$ if f is continuous in $0 < x < \infty$ and if

$$\sum_{n=-\infty}^{\infty} \max_{2^n \leq x \leq 2^{n+1}} |xf(x)| < \infty.$$

For such functions we adopt the notation

$$(3.10) \quad F(\mu) = \int_0^{\infty} f(x)x^{i\mu}dx, \quad F = F(0),$$

where μ is real. We say that $\alpha \in V$ if α has bounded variation over each finite interval in $0 < x < \infty$ and if

$$\int_x^{2x} \left| \frac{d\alpha(y)}{y} \right|$$

is bounded for⁽²⁾ $0 < x < \infty$.

We have next the following variant of the Wiener Tauberian theorem [6, Corollary 1].

LEMMA 3. Suppose that $f_1, f_2, g_1, g_2 \in M$, that $\alpha, \beta \in V$, that α, β are monotone, that $\alpha(0) = \beta(0) = 0$, and that

$$(3.11) \quad F_1(\mu)G_2(\mu) - F_2(\mu)G_1(\mu) \neq 0$$

for all real μ . If

$$\begin{aligned} \frac{1}{s} \int_0^{\infty} f_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} f_2(y/s) d\beta(y) &\rightarrow A, \\ \frac{1}{s} \int_0^{\infty} g_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} g_2(y/s) d\beta(y) &\rightarrow B \end{aligned}$$

as $s \rightarrow 0 + (\infty)$, then

$$\frac{\alpha(x)}{x} \rightarrow \frac{AG_2 - BF_2}{F_1G_2 - F_2G_1}, \quad \frac{\beta(x)}{x} \rightarrow \frac{BF_1 - AG_1}{F_1G_2 - F_2G_1}$$

as $x \rightarrow 0 + (\infty)$.

In the proof of Theorems 5 and 6 we use the following result to show that condition (3.11) is satisfied for a special set of functions.

LEMMA 4. For real x and μ let

$$\begin{aligned} \theta_1(x) &= \theta_1(x, \mu) = \int_0^{\infty} e^{-(v-x)^2} y^{i\mu} dy, \\ \theta_2(x) &= \theta_2(x, \mu) = \int_0^{\infty} e^{-(v+x)^2} y^{i\mu} dy. \end{aligned}$$

⁽²⁾ If $f \in M$ and $\alpha \in V$, then the integral $\int_0^{\infty} f(x) d\alpha(x)$ is absolutely convergent. In particular if $f \in M$, then f is Lebesgue integrable over $0 < x < \infty$.

Then, for each fixed μ ,

$$(3.12) \quad \theta_1(a)\theta_2(b) - \theta_2(a)\theta_1(b) \neq 0$$

for all a and b , $a \neq b$, and

$$(3.13) \quad \theta_1(a)\theta_2'(b) - \theta_2(a)\theta_1'(b) \neq 0$$

or all a and b .

Proof. If $w=w(x)$ is any solution of the differential equation

$$(3.14) \quad w'' + 2xw' - 2i\mu w = 0,$$

then a standard calculation yields

$$e^{x^2} \bar{w} w' \Big|_a^b = \int_a^b e^{x^2} |w'|^2 dx + 2i\mu \int_a^b e^{x^2} |w|^2 dx$$

for real a and b , where the bar denotes complex conjugate. (Cf. [10, p. 509].) If w is a nontrivial solution of (3.14) with $w(a)=0$, then

$$R\{\bar{w}(b)w'(b)\} = e^{-b^2} \int_a^b e^{x^2} |w'|^2 dx > 0,$$

where $R(z)$ denotes the real part of the complex number z . We conclude that w has a as its only real root and that w' has no real roots at all.

Now suppose that (3.12) does not hold for some $a \neq b$. Then we can find a nontrivial pair of constants A and B such that

$$(3.15) \quad w(x) = A\theta_1(x) + B\theta_2(x)$$

has real roots at a and b . By differentiating under the integral sign it is easy to verify that w , as defined in (3.15), is a solution of (3.14). Hence, by the previous argument, $w \equiv 0$. But this is clearly impossible since, for $x=0$, the Wronskian of θ_1 and θ_2 is simply

$$-2\theta_1(0)\theta_1'(0) = -\Gamma\left(\frac{1+i\mu}{2}\right)\Gamma\left(1+\frac{i\mu}{2}\right) \neq 0.$$

Similarly if (3.13) does not hold we can find a nontrivial pair of constants A and B such that w , as defined in (3.15), has a root at a and such that w' has a root at b . Again this implies that $w \equiv 0$ and we obtain a contradiction as before.

Finally we require the following partial converse for Corollary 1.

LEMMA 5. Suppose that $u \in H^+$ in $0 < t < c$. If, for some path $\gamma \subset P(a, x_0)$,

$$(3.16) \quad u(x, t) = O(1)$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in \gamma$, then

$$(3.17) \quad \frac{\alpha(x_0 + x) - \alpha(x_0)}{x} = O(1)$$

as $x \rightarrow 0$. Suppose that $u \in H^+$ in $0 < t < \infty$. If, for some path $\gamma \subset P(a, x_0)$, (3.16) holds as $t \rightarrow \infty$, $(x, t) \in \gamma$, then (3.17) holds as $|x| \rightarrow \infty$.

Proof. Assume $x_0 = 0$. Then $(x, t) \in P(a, 0)$ and $|y| \leq t^{1/2}$ imply that

$$k(x - y, t) \geq \frac{1}{(4\pi t)^{1/2}} e^{-(1+a)^2/4} = \frac{C_4}{2t^{1/2}}$$

where $C_4 = C_4(a) > 0$. Since $u \in H^+$, α is nondecreasing and we obtain

$$(3.18) \quad 0 \leq \frac{\alpha(t^{1/2}) - \alpha(-t^{1/2})}{2t^{1/2}} \leq \frac{1}{C_4} \int_{-t^{1/2}}^{t^{1/2}} k(x - y, t) d\alpha(y) \leq \frac{u(x, t)}{C_4}$$

for $(x, t) \in \gamma \subset P(a, 0)$. Both parts of the lemma follow from (3.16) and (3.18).

We consider now the proofs for Theorem 5 and Theorem 6.

Proof of Theorem 5. We begin with the first part of Theorem 5. Let $\alpha(x_0) = 0$, $x_0 = 0$, and, for convenience, replace a by $2a$ and b by $2b$ in (3.7). Then we can write

$$\begin{aligned} u(2at^{1/2}, t) &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-[(y/2t)^2 - a]^2/4} d\alpha(y) = \frac{1}{s} \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-[(y/s) - a]^2} d\alpha(y) \\ &= \frac{1}{s} \int_0^{\infty} f_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} f_2(y/s) d\beta(y), \end{aligned}$$

where $s = 2t^{1/2}$, $\beta(x) = -\alpha(-x)$, and

$$f_1(x) = \frac{1}{\pi^{1/2}} e^{-(x-a)^2}, \quad f_2(x) = \frac{1}{\pi^{1/2}} e^{-(x+a)^2}.$$

Similarly, we can write

$$u(2bt^{1/2}, t) = \frac{1}{s} \int_0^{\infty} g_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} g_2(y/s) d\beta(y),$$

where s and β are as above and where

$$g_1(x) = \frac{1}{\pi^{1/2}} e^{-(x-b)^2}, \quad g_2(x) = \frac{1}{\pi^{1/2}} e^{-(x+b)^2}.$$

The hypotheses (3.7) now imply that

$$(3.19) \quad \begin{aligned} \frac{1}{s} \int_0^{\infty} f_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} f_2(y/s) d\beta(y) &\rightarrow A, \\ \frac{1}{s} \int_0^{\infty} g_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^{\infty} g_2(y/s) d\beta(y) &\rightarrow A \end{aligned}$$

as $s \rightarrow 0+$. To complete the proof we will show that Lemma 3 can be applied to (3.19).

Since f_1, f_2, g_1, g_2 are continuous in $0 \leq x < \infty$ and are $O(1/x^2)$ for large x , it follows that all of these functions are in M [8, p. 299]. Obviously α and β are monotone and $\alpha(0) = \beta(0) = 0$. From (3.7) and Lemma 5 we obtain

$$(3.20) \quad \begin{aligned} \int_x^{2x} \frac{|d\alpha(y)|}{y} &\leq \frac{\alpha(2x) - \alpha(x)}{x} = O(1), \\ \int_x^{2x} \frac{|d\beta(y)|}{y} &\leq \frac{\alpha(-x) - \alpha(-2x)}{x} = O(1) \end{aligned}$$

for $0 < x \leq d$, and, appealing to (2.2), we see we can assume that (3.20) holds for $0 < x < \infty$. Hence $\alpha, \beta \in V$. Using (3.10),

$$\begin{aligned} F_1(\mu) &= \frac{1}{\pi^{1/2}} \theta_1(a), & F_2(\mu) &= \frac{1}{\pi^{1/2}} \theta_2(a), \\ G_1(\mu) &= \frac{1}{\pi^{1/2}} \theta_1(b), & G_2(\mu) &= \frac{1}{\pi^{1/2}} \theta_2(b), \end{aligned}$$

for each real μ , where θ_1 and θ_2 are as defined in Lemma 4, and hence

$$F_1(\mu)G_2(\mu) - F_2(\mu)G_1(\mu) \neq 0$$

by (3.12). Finally since

$$F_1G_2 - F_2G_1 = F_1 - G_1 = G_2 - F_2,$$

we can apply Lemma 3 to (3.19) to conclude that

$$(3.21) \quad \frac{\alpha(x)}{x} \rightarrow A, \quad \frac{-\alpha(-x)}{x} = \frac{\beta(x)}{x} \rightarrow A$$

as $x \rightarrow 0+$, and hence that $\alpha'(0) = A$.

For the second part of Theorem 5 proceed as before. The hypothesis (3.7) now implies that (3.19) holds as $s \rightarrow \infty$, and, with Lemma 5, we obtain (3.20) for $d \leq x < \infty$. Then (2.3) allows us to assume that (3.20) is valid for $0 < x < \infty$ and again $\alpha, \beta \in V$. The proof is then completed by appealing to the $s, x \rightarrow 0$ part of Lemma 3.

Proof for Theorem 6. Let $\alpha(x_0) = 0, x_0 = 0$, and replace a by $2a$ and b by $2b$ in (3.9). Then arguing as before and differentiating under the integral sign in (1.1) we obtain

$$\begin{aligned} u(2at^{1/2}, t) &= \frac{1}{s} \int_0^\infty f_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^\infty f_2(y/s) d\beta(y), \\ t^{1/2}u_x(2bt^{1/2}, t) &= \frac{1}{s} \int_0^\infty g_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^\infty g_2(y/s) d\beta(y), \end{aligned}$$

where $s = 2t^{1/2}$, $\beta(x) = -\alpha(x)$, and where

$$\begin{aligned} f_1(x) &= \frac{1}{\pi^{1/2}} e^{-(x-a)^2}, & f_2(x) &= \frac{1}{\pi^{1/2}} e^{-(x+a)^2}, \\ g_1(x) &= \frac{x-b}{\pi^{1/2}} e^{-(x-b)^2}, & g_2(x) &= -\frac{x+b}{\pi^{1/2}} e^{-(x+b)^2}. \end{aligned}$$

The hypotheses for the first part of Theorem 6 then imply that

$$\begin{aligned} (3.22) \quad & \frac{1}{s} \int_0^\infty f_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^\infty f_2(y/s) d\beta(y) \rightarrow A, \\ & \frac{1}{s} \int_0^\infty g_1(y/s) d\alpha(y) + \frac{1}{s} \int_0^\infty g_2(y/s) d\beta(y) \rightarrow 0 \end{aligned}$$

as $s \rightarrow 0+$. Now f_1, f_2, g_1, g_2 are in M , α and β are monotone with $\alpha(0) = \beta(0) = 0$, and the argument of Theorem 5 shows we can assume that α and β are in V . Since

$$\begin{aligned} F_1(\mu) &= \frac{1}{\pi^{1/2}} \theta_1(a), & F_2(\mu) &= \frac{1}{\pi^{1/2}} \theta_2(a), \\ G_1(\mu) &= \frac{1}{2\pi^{1/2}} \theta'_1(b), & G_2(\mu) &= \frac{1}{2\pi^{1/2}} \theta'_2(b), \end{aligned}$$

we see from (3.13) that

$$F_1(\mu)G_2(\mu) - F_2(\mu)G_1(\mu) \neq 0$$

for all real μ . Hence we can apply Lemma 3 to (3.22) and, since

$$F_1G_2 - F_2G_1 = G_2 = -G_1,$$

we conclude that (3.21) holds as $x \rightarrow 0+$. This completes the proof for the first part of Theorem 6. The proof for the second part follows similarly.

4. Additional remarks. In the first part of Theorem 4 we ask that $u \rightarrow A$ as $(x, t) \rightarrow (x_0, 0)$ along one ray. In the first part of Theorem 5 we ask that $u \rightarrow A$ as $(x, t) \rightarrow (x_0, 0)$ along two different parabolic arcs, i.e. the curves

$$x = x_0 + at^{1/2}, \quad x = x_0 + bt^{1/2}, \quad a \neq b.$$

The following example shows that two arcs are required in the hypotheses of Theorem 5.

THEOREM 7. *Given any fixed number a , there exists a bounded $u \in H^+$ in $0 < t < \infty$ such that*

$$u(at^{1/2}, t) = 1$$

for $0 < t < \infty$ and such that, for each $b \neq a$,

$$\liminf u(bt^{1/2}, t) < \limsup u(bt^{1/2}, t)$$

as $t \rightarrow 0+$ and as $t \rightarrow \infty$.

Proof. For convenience replace a by $2a$ and b by $2b$. Then let f_1, f_2, g_1, g_2 be as defined in the proof for Theorem 5 and, using (3.10), let

$$\begin{aligned}\alpha(x) &= x + R \left\{ F_2(\mu) \frac{x^{i\mu+1}}{i\mu + 1} \right\}, \\ -\alpha(-x) &= x - R \left\{ F_1(\mu) \frac{x^{i\mu+1}}{i\mu + 1} \right\}\end{aligned}$$

for $0 \leq x < \infty$, where $\mu \neq 0$ and where $R(z)$ again denotes the real part of the complex number z . Since

$$\begin{aligned}|F_1(\mu)| &\leq \frac{1}{\pi^{1/2}} \int_0^\infty e^{-(x-a)^2} dx < 1, \\ |F_2(\mu)| &\leq \frac{1}{\pi^{1/2}} \int_0^\infty e^{-(x+a)^2} dx < 1,\end{aligned}$$

we see that $0 < \alpha'(x) < 2$ for $x \neq 0$, and hence that

$$u(x, t) = \int_{-\infty}^\infty k(x - y, t) d\alpha(y) = \int_{-\infty}^\infty k(x - y, t) \alpha'(y) dy$$

is bounded and in H^+ over $0 < t < \infty$. Following the proof for Theorem 5 we have

$$\begin{aligned}u(2at^{1/2}, t) &= \frac{1}{s} \int_0^\infty f_1(y/s) d\alpha(y) - \frac{1}{s} \int_0^\infty f_2(y/s) d\alpha(-y) \\ &= 1 + R \left\{ \frac{F_2(\mu)}{s} \int_0^\infty f_1(y/s) y^{i\mu} dy - \frac{F_1(\mu)}{s} \int_0^\infty f_2(y/s) y^{i\mu} dy \right\} \\ &= 1 + R \{ s^{i\mu} [F_2(\mu) F_1(\mu) - F_1(\mu) F_2(\mu)] \} = 1\end{aligned}$$

for $0 < t < \infty$, $s = 2t^{1/2}$. Similarly

$$\begin{aligned}u(2bt^{1/2}, t) &= \frac{1}{s} \int_0^\infty g_1(y/s) d\alpha(y) - \frac{1}{s} \int_0^\infty g_2(y/s) d\alpha(-y) \\ &= 1 + R \{ s^{i\mu} [F_2(\mu) G_1(\mu) - F_1(\mu) G_2(\mu)] \}\end{aligned}$$

for $0 < t < \infty$. Since $b \neq a$, (3.12) yields

$$H(\mu) = F_2(\mu) G_1(\mu) - F_1(\mu) G_2(\mu) \neq 0,$$

and we conclude that

$$\limsup u(2bt^{1/2}, t) - \liminf u(2bt^{1/2}, t) = 2|H(\mu)| > 0$$

as $t \rightarrow 0+$ and as $t \rightarrow \infty$. This completes the proof for Theorem 7.

It is clear, in the above example, that $\alpha'(0)$ does not exist and that $\alpha(x)/x$ does not have a limit as $|x| \rightarrow \infty$. Moreover, if we fix $a \neq 0$, it is easy to see that $D\alpha(0)$ does not exist and that

$$\frac{\alpha(x) - \alpha(-x)}{2x}$$

does not have a limit as $x \rightarrow \infty$. Hence we conclude that, when $a \neq 0$, the hypothesis

$$u(x_0 + at, t) \rightarrow A \text{ as } t \rightarrow 0+,$$

in the first part of Theorem 4, cannot be replaced by the hypothesis

$$u(x_0 + at^{1/2}, t) \rightarrow A \text{ as } t \rightarrow 0+.$$

When E is any set in the xt -plane, we let E_c denote the part of E which is contained in the half plane $t \leq c$. If we make use of the following maximum principle (see [7] or [17]) for functions in H , we can replace the two parabolic arcs in Theorem 5 by more general curves.

LEMMA 6. *Suppose that $u \in H$ in a bounded domain δ with boundary γ . Given any c , if*

$$\limsup_{(x,t) \rightarrow (x_1,t_1)} u(x,t) \leq A \quad (x,t) \in \delta_c,$$

for all $(x_1, t_1) \in \gamma_c$, then $u \leq A$ in δ_c .

Here, and in what follows, we let $D = D(x_0)$ denote an infinite domain bounded by two disjoint arcs in $0 < t < \infty$ which terminate at the point $(x_0, 0)$ and which cross the line $t = c$ for each $0 < c < \infty$. We assume further that D_c is bounded for each such c and we let $\Gamma = \Gamma(x_0)$ denote the boundary for D .

THEOREM 8. *Suppose that $u \in H$ in $0 < t < c$ and that, for some $0 < b < c$, u is bounded in D_b . If*

$$(4.1) \quad u(x, t) \rightarrow A$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in \Gamma$, then (4.1) holds as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in D$. Suppose that $u \in H$ in $0 < t < \infty$ and that, for some $b > 0$, u is bounded in $D - D_b$. If (4.1) holds as $t \rightarrow \infty$, $(x, t) \in \Gamma$, then (4.1) holds as $t \rightarrow \infty$, $(x, t) \in D$.

Proof. Consider the first part. For each $\epsilon > 0$ we can find $a > 0$ such that $|u - A| < M$ in D_a and such that

$$(4.2) \quad |u(x, t) - A| \leq \epsilon$$

for $(x, t) \in \Gamma_a$, $(x, t) \neq (x_0, 0)$. Next for each $d > 0$ let

$$v(x, t) = u(x, t) - A - M \int_{|y-x_0| \leq d} k(x-y, t) dy.$$

Clearly $v \in H$ in $0 < t < c$. From (2.2) of Lemma 1 we see that

$$\int_{|y-x_0| \leq d} k(x-y, t) dy \rightarrow 1$$

as $(x, t) \rightarrow (x_0, 0)$, $t > 0$, and, with (4.2), we obtain

$$\limsup_{(x, t) \rightarrow (x_1, t_1)} v(x, t) \leq \epsilon \quad (x, t) \in D_a,$$

for all $(x_1, t_1) \in \Gamma_a$. Applying Lemma 6 to v yields

$$u(x, t) \leq A + \epsilon + M \int_{|y-x_0| \leq d} k(x-y, t) dy$$

for $(x, t) \in D_a$, and letting $d \rightarrow 0+$ we obtain $u \leq A + \epsilon$ in D_a . A similar argument gives $u \geq A - \epsilon$ in D_a thus completing the proof for the first part of Theorem 8.

For the second part pick $a > 0$ so that $|u - A| < M$ in $D - D_a$ and so that (4.2) holds for all $(x, t) \in \Gamma - \Gamma_a$. Let γ denote the boundary for $D - D_a$. Since D_a is bounded we can find a $d > 0$ such that γ_a , the part of γ contained in the line $t = a$, lies between the lines $x = \pm d$. Now set

$$v(x, t) = u(x, t) - A - M \int_{|y| \leq d} k(x-y, t-a) dy.$$

Then $v \in H$ in $a < t < \infty$ and, again with (2.2), we see that for each $|x_1| < d$,

$$\int_{|y| \leq d} k(x-y, t-a) dy \rightarrow 1$$

as $(x, t) \rightarrow (x_1, a)$, $t > a$. Since $\gamma = \gamma_a \cup (\Gamma - \Gamma_a)$ we conclude that

$$\limsup_{(x, t) \rightarrow (x_1, t_1)} v(x, t) \leq \epsilon \quad (x, t) \in D - D_a,$$

for all $(x_1, t_1) \in \gamma$. If we apply Lemma 6 to v over $D_c - D_a$, $a < c < \infty$, and then let $c \rightarrow \infty$, we obtain

$$u(x, t) \leq A + \epsilon + M \int_{|y| \leq d} k(x-y, t-a) dy$$

for $(x, t) \in D - D_a$. A similar argument yields

$$u(x, t) \geq A - \epsilon - M \int_{|y| \leq d} k(x-y, t-a) dy$$

for $(x, t) \in D - D_a$ and since, by (2.4),

$$\left| \int_{|y| \leq d} k(x-y, t-a) dy \right| \leq \frac{d}{(\pi(t-a))^{1/2}} \rightarrow 0$$

as $t \rightarrow \infty$, uniformly in x , $|u-A| \leq 2\epsilon$ in $D-D_c$ for some $c > a$. This completes the proof for the second part of Theorem 8.

The Phragmén-Lindelöf type argument used in the proof of the first part of Theorem 8 can be combined with a familiar step-by-step argument to give the following variant of Lemma 6.

COROLLARY 3. *Suppose that u is bounded above and that $u \in H$ in a bounded domain δ with boundary γ . Given any c , if*

$$\limsup_{(x,t) \rightarrow (x_1,t_1)} u(x,t) \leq A \quad (x,t) \in \delta_c,$$

for all but a finite set of $(x_1, t_1) \in \gamma_c$, then $u \leq A$ in δ_c .

We can now prove the following extension of Theorem 5.

THEOREM 9. *Suppose that $D(x_0) \subset P(a, x_0)$, for some $a > 0$, and that D contains two parabolic arcs through $(x_0, 0)$. If $u \in H^+$ in $0 < t < c$ and if*

$$(4.3) \quad u(x, t) \rightarrow A$$

as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in \Gamma$, then $\alpha'(x_0) = A$. If $u \in H^+$ in $0 < t < \infty$ and if (4.3) holds as $t \rightarrow \infty$, $(x, t) \in \Gamma$, then

$$\frac{\alpha(x_0 + x) - \alpha(x_0)}{x} \rightarrow A$$

as $|x| \rightarrow \infty$.

Proof. The hypotheses for the first part of Theorem 9 imply that

$$(4.4) \quad u(x, t) = O(1)$$

as $(x, t) \rightarrow (x_0, 0)$ along some path in $P(a, x_0)$. From Lemma 5 and Corollary 1 we conclude that (4.4) holds as $(x, t) \rightarrow (x_0, 0)$, $(x, t) \in P(a, x_0)$, and hence that u is bounded in D_b for some $0 < b < c$. With Theorem 8 we see that (4.3) holds as $(x, t) \rightarrow (x_0, 0)$ along the two parabolic arcs in D and the desired conclusion follows from Theorem 5. The proof for the second part is very similar.

COROLLARY 4. *If, in Theorem 9, u is bounded, we can drop the restriction that $D(x_0)$ lie in some $P(a, x_0)$.*

This is an immediate consequence of the above argument since this restriction is used only in proving that u is bounded in D_b or in $D-D_b$. On the other hand, without boundedness it is clear that some such restriction is

necessary. For if we let D be the domain bounded by the curve $x^4 = t$ and set $u(x, t) = k(x, t)$, then

$$u(x, t) \rightarrow 0$$

as $(x, t) \rightarrow (0, 0)$, $(x, t) \in \Gamma$. But the normalized α is constant except for a jump of 1 at $x = 0$ and $\alpha'(0) = \infty$.

We conclude this section with a pair of remarks on steady state temperatures which are trivial consequences of some of the preceding results.

COROLLARY 5. Suppose that $u \in H^+$ in $0 < t < \infty$. If, for some x_0 ,

$$(4.5) \quad u(x_0, t) \rightarrow A$$

as $t \rightarrow \infty$, then, for each $a > 0$,

$$(4.6) \quad u(x, t) \rightarrow A$$

as $t \rightarrow \infty$, uniformly in x in $|x - x_0| < a$.

COROLLARY 6. Suppose that $u \in H^+$ in $0 < t < \infty$ and that, for some $0 < c < \infty$, $u(x, c)$ is monotone in x . If, for some x_0 ,

$$(4.7) \quad u(x_0, t) = O(1)$$

as $t \rightarrow \infty$, then there exists a constant A such that, for each $a > 0$, (4.6) holds as $t \rightarrow \infty$, uniformly in x in $|x - x_0| < a$.

Proof for Corollary 5. From (4.5), Lemma 5, and Corollary 1 we see that, for each $a > 0$,

$$u_x(x, t) = O\left(\frac{1}{t^{1/2}}\right)$$

as $t \rightarrow \infty$, $(x, t) \in P(a, x_0)$. Thus when $|x - x_0| < a$, $t > 1$, we have

$$|u(x, t) - u(x_0, t)| \leq \frac{C_5}{t^{1/2}},$$

where $C_5 = C_5(a, u)$, and the desired conclusion follows from (4.5).

Proof for Corollary 6. Let $x_0 = 0$ and set $v(x, t) = u(x, t + c)$. Then $v \in H^+$ in $0 < t < \infty$ and, with (1.3),

$$v(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\beta(y), \quad \beta(x) = \int_0^x u(y, c) dy.$$

If $u(x, c)$ is nondecreasing or nonincreasing in x , it follows that $\beta(x)/x$ is nondecreasing or nonincreasing, respectively, in x , $x \neq 0$. Next (4.7) and Lemma 5 imply that $\beta(x)/x$ is bounded as $|x| \rightarrow \infty$. We conclude that

$$\frac{\beta(x) - \beta(-x)}{2x}$$

has a finite limit A as $x \rightarrow \infty$ and hence, with Theorem 1, that

$$u(0, t) \rightarrow A$$

as $t \rightarrow \infty$. The rest follows from Corollary 5.

In Corollary 6 we replace (4.5) by the weaker (4.7) and ask that the temperature be monotone at some time $t=c$. Some such additional condition is necessary. For let u be the bounded function of Theorem 7 with $a \neq 0$ and $b=0$. Then (4.7) is satisfied,

$$\liminf_{t \rightarrow \infty} u(0, t) < \limsup_{t \rightarrow \infty} u(0, t),$$

and, with Corollary 5, we see that

$$\liminf_{t \rightarrow \infty} u(x_0, t) < \limsup_{t \rightarrow \infty} u(x_0, t)$$

for all x_0 .

5. Uniqueness theorems. We consider here some one-sided uniqueness theorems for functions in H over $0 < t < c$. We begin with the following preliminary result.

LEMMA 7. *Suppose that α has bounded variation over each finite interval, that*

$$(5.1) \quad \alpha(x-0) \geq \alpha(x) \geq \alpha(x+0)$$

for all x , and that $\alpha'(x) < \infty$ for all x , except an enumerable set, at which the derivative exists. Then

$$\alpha(x_2) - \alpha(x_1) \leq \int_{x_1}^{x_2} \alpha'(x) dx$$

for all $x_1, x_2, x_1 < x_2$.

Proof. This is an immediate consequence of the de la Vallée Poussin decomposition theorem [16, p. 127]. For let I denote the open interval $x_1 < x < x_2$, let X denote the points in I where α is continuous, and let

$$E_{\infty} \equiv \{x: \alpha'(x) = \infty, x \in X\}, \quad E_{-\infty} \equiv \{x: \alpha'(x) = -\infty, x \in X\}.$$

If $\mu = \mu(E)$ denotes the signed measure corresponding to α , then the de la Vallée Poussin theorem yields

$$(5.2) \quad \mu(X) = \mu(E_{\infty}) + \mu(E_{-\infty}) + \int_X \alpha'(x) dx.$$

By hypothesis E_{∞} is at most enumerable and hence $\mu(E_{\infty}) = 0$. Clearly $\mu(E_{-\infty}) \leq 0$ and, from (5.1), we see that

$$(5.3) \quad \mu(I) = \mu(I - X) + \mu(X) \leq \mu(X).$$

Since $I - X$ is enumerable we can combine (5.1), (5.2), and (5.3) to obtain

$$\alpha(x_2) - \alpha(x_1) \leq \mu(I) \leq \int_{x_1}^{x_2} \alpha'(x) dx$$

as desired.

The following theorem extends results of Rosenbloom [15] and Widder [19] and is a one-sided analogue of a theorem on harmonic functions due to Lohwater [11].

THEOREM 10. *Suppose that $u \in H^\Delta$ in $0 < t < c$. If*

$$(5.4) \quad \lim_{t \rightarrow 0+} u(x, t) < \infty$$

for all x at which this limit exists and if⁽³⁾

$$(5.5) \quad \lim_{t \rightarrow 0+} u(x, t) \leq A$$

for almost all x , then $u \leq A$ in $0 < t < c$.

Proof. If α has a jump of λ at $x = x_0$, then, by Lemma 2,

$$u(x_0, t) \sim \frac{\lambda}{(4\pi t)^{1/2}}$$

as $t \rightarrow 0+$, and, with (5.4), it follows that $\lambda < 0$. Hence, assuming that α is normalized, (5.1) holds for all x . Next suppose that α has a derivative, finite or infinite, at x_0 . Then, with Theorem 2 and Corollary 2, we see that

$$\lim_{t \rightarrow 0+} u(x_0, t) = \alpha'(x_0),$$

and we conclude from (5.4) that $\alpha'(x) < \infty$ for all x at which this derivative exists. From (5.5) we obtain $\alpha'(x) \leq A$ for almost all x . Lemma 7 yields

$$\alpha(x_2) - \alpha(x_1) \leq \int_{x_1}^{x_2} \alpha'(x) dx \leq A(x_2 - x_1)$$

for all x_1, x_2 , $x_1 < x_2$, and we conclude that

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y) \leq A \int_{-\infty}^{\infty} k(x - y, t) dy = A$$

in $0 < t < c$.

The following is an immediate consequence of Theorem 10.

COROLLARY 7. *Suppose that $u \in H^\Delta$ in $0 < t < c$. If*

$$\lim_{t \rightarrow 0+} u(x, t) = A$$

⁽³⁾ Since α has a finite derivative almost everywhere, the limit in (5.5) exists for almost all x by virtue of Theorem 2.

for all x , then $u \equiv A$ in $0 < t < c$.

Theorem 10 can also be used to obtain the following one-sided version of a well known theorem due to Tychonoff [18].

THEOREM 11. *Suppose that $u \in H$ in $0 < t < c$. If*

$$(5.6) \quad u(x, t) \leq M e^{ax^2}, \quad M > 0, a > 0,$$

for all x , $0 < t < c$, and if

$$(5.7) \quad \liminf_{t \rightarrow 0+} u(x, t) \leq A$$

for almost all x , then $u \leq A$ in $0 < t < c$.

Proof. Let $b = \text{Min}(1/4a, c)$ and let

$$(5.8) \quad v(x, t) = \frac{M}{(1 - 4at)^{1/2}} e^{ax^2/(1-4at)}.$$

Then $v, v - u \in H^+$ in $0 < t < b$. Hence $u \in H^A$ in this strip and, with (5.7),

$$\lim_{t \rightarrow 0+} u(x, t) = \liminf_{t \rightarrow 0+} u(x, t) \leq A$$

for almost all x . From (5.6) we see that

$$\lim_{t \rightarrow 0+} u(x, t) \leq M e^{ax^2} < \infty$$

for all x at which this limit exists and, appealing to Theorem 10, we conclude that $u \leq A$ in $0 < t < b$. If $b < c$, the proof is completed by a familiar step-by-step argument [18].

COROLLARY 8. *Suppose that $u \in H$ in $0 < t < c$. If (5.6) holds for all x , $0 < t < c$, and if*

$$\lim_{t \rightarrow 0+} u(x, t) = A$$

for all x , then $u \equiv A$ in $0 < t < c$.

Proof. With Theorem 11 we see that $u \leq A$ in $0 < t < c$. Hence $A - u \in H^+$ and $u \in H^A$ in this strip, and Corollary 7 yields $u \equiv A$.

We have next the following one-sided version of a recent theorem due to Birkhoff and Kotik [1].

THEOREM 12. *Suppose that $u \in H$ in $0 < t < c$. If*

$$(5.9) \quad \int_{x_1}^{x_2} u(y, t) dy \leq M e^{a(x_1^2 + x_2^2)}, \quad M > 0, a > 0,$$

for all $x_1, x_2, x_1 < x_2, 0 < t < c$, and if

$$(5.10) \quad \liminf_{t \rightarrow 0+} \int_{x_1}^{x_2} u(y, t) dy \leq A(x_2 - x_1)$$

for almost all x_1, x_2 , $x_1 < x_2$, then $u \leq A$ in $0 < t < c$.

Proof. By the last hypothesis we mean that (5.10) holds for all $x_1, x_2 \notin E$, $x_1 < x_2$, where E is a set of measure zero in the x -axis. For each $h > 0$ let

$$v_h(x, t) = \frac{1}{2h} \int_{-h}^h u(x + y, t) dy.$$

Then $v_h \in H$ in $0 < t < c$ and, from (5.9), we see that

$$v_h(x, t) \leq N e^{bx^2}$$

for all x , $0 < t < c$, where $N = (1/2h) M e^{2ah^2}$, $b = 2a$. From (5.10) it follows that

$$\liminf_{t \rightarrow 0+} v_h(x, t) \leq A$$

for almost all x and, appealing to Theorem 11, we conclude that $v_h \leq A$ in $0 < t < c$. Finally

$$u(x, t) = \lim_{h \rightarrow 0+} v_h(x, t) \leq A$$

in $0 < t < c$ as desired.

With the aid of Theorem 12 we obtain the following extension of the above mentioned theorem of Birkhoff and Kotik.

THEOREM 13. Suppose that $u \in H$ in $0 < t < c$. If (5.9) holds for all x_1, x_2 , $x_1 < x_2$, $0 < t < c$, and if

$$(5.11) \quad \lim_{t \rightarrow 0+} \int_{x_1}^{x_2} u(y, t) dy = A(x_2 - x_1)$$

for almost all x_1, x_2 , then $u \equiv A$ in $0 < t < c$.

In the Birkhoff-Kotik theorem, the one-sided restriction (5.9) is replaced by a two-sided condition equivalent to the following:

$$\left| \int_{x_1}^{x_2} u(y, t) dy \right| \leq M e^{a(x_1^2 + x_2^2)}, \quad M > 0, a > 0,$$

for all x_1, x_2 , $0 < t < c$.

Proof. By Theorem 12 we see that $u \leq A$ in $0 < t < c$ and hence that $u \in H^\Delta$ in this strip. Next if we assume, as we may, that α is normalized, then (1.3) and (5.11) yield

$$(5.12) \quad \alpha(x_2) - \alpha(x_1) = \lim_{t \rightarrow 0+} \int_{x_1}^{x_2} u(y, t) dy = A(x_2 - x_1)$$

for almost all x_1, x_2 . Since (5.12) holds for all $x_1, x_2 \in E$, where E is dense in the x -axis, we conclude from a limiting argument that

$$\alpha(x_2) - \alpha(x_1) = A(x_2 - x_1)$$

for all x_1, x_2 , and hence that $u \equiv A$ in $0 < t < c$.

The following is an immediate consequence of the above argument.

COROLLARY 9. *Suppose that $u \in H^\Delta$ in $0 < t < c$ and that E is a set dense in the x -axis. If*

$$\lim_{t \rightarrow 0+} \int_{x_1}^{x_2} u(y, t) dy \leq A(x_2 - x_1)$$

for all $x_1, x_2 \in E$, $x_1 < x_2$, then $u \leq A$ in $0 < t < c$.

We require next the following variant of the parabolic maximum principle, Lemma 6.

LEMMA 8. *Suppose that $u \in H$ in $0 < t < c$ and that δ is the rectangle bounded by the lines $x = x_1, x = x_2, t = 0, t = c$, where $x_1 < x_2$. If*

$$(5.13) \quad u(x_1, t) \leq A, \quad u(x_2, t) \leq A$$

for $0 < t < c$, and if, for each non-negative continuous function $f(x)$,

$$(5.14) \quad \liminf_{t \rightarrow 0+} \int_{x_1}^{x_2} f(y) u(y, t) dy \leq A \int_{x_1}^{x_2} f(y) dy,$$

then $u \leq A$ in δ .

Proof. We can obviously assume that $A = 0$. Next fix $(x, t) \in \delta$ and pick $\epsilon > 0$ so that $0 < t + \epsilon < c$. Then we can write

$$\begin{aligned} u(x, t + \epsilon) = & \int_{x_1}^{x_2} G(x, t; y, 0) u(y, \epsilon) dy + \int_0^t \frac{\partial G}{\partial y} (x, t; x_1, s) u(x_1, s + \epsilon) ds \\ & - \int_0^t \frac{\partial G}{\partial y} (x, t; x_2, s) u(x_2, s + \epsilon) ds, \end{aligned}$$

where G is the Green's function

$$G(x, t; y, s) = \frac{1}{2l} \left\{ \theta_3 \left(\frac{x-y}{2l}, \frac{t-s}{l^2} \right) - \theta_3 \left(\frac{x+y-2x_1}{2l}, \frac{t-s}{l^2} \right) \right\},$$

where $l = x_2 - x_1$, and where θ_3 is the Jacobi theta function

$$\theta_3(x, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \cos 2n\pi x.$$

(See [5] or [9] and make a change of variables.) Since

$$\frac{\partial G}{\partial y}(x, t; x_1, s) \geq 0, \quad -\frac{\partial G}{\partial y}(x, t; x_2, s) \geq 0,$$

we see from (5.13) that

$$u(x, t + \epsilon) \leq \int_{x_1}^{x_2} G(x, t; y, 0)u(y, \epsilon)dy$$

and, letting $\epsilon \rightarrow 0+$, we conclude from (5.14) that

$$u(x, t) \leq \liminf_{\epsilon \rightarrow 0+} \int_{x_1}^{x_2} G(x, t; y, 0)u(y, \epsilon)dy \leq 0$$

as desired.

We consider next the following extension of a uniqueness theorem due to Cooper [2].

THEOREM 14. *Suppose that $u \in H$ in $0 < t < c$ and that $\{x_n\}$ is a sequence, defined for $-\infty < n < \infty$, such that $x_n \rightarrow \infty (-\infty)$ as $n \rightarrow \infty (-\infty)$. If*

$$(5.15) \quad u(x_n, t) \leq Me^{ax_n^2}, \quad M > 0, a > 0,$$

for all n , $0 < t < c$, and if u converges weakly to A , over each finite interval, as $t \rightarrow 0+$, then $u \equiv A$ in $0 < t < c$.

In Cooper's theorem, the one-sided restriction (5.15) is replaced by a two-sided condition, namely that

$$|u(x_n, t)| \leq Me^{ax_n^2}, \quad M > 0, a > 0,$$

for all n , $0 < t < c$.

Proof. Again we can assume that $A = 0$. Next let $b = \text{Min}(1/4a, c)$, define v as in (5.8), and let $w = u - v$. If we pick $x_m < x_n$, then

$$(5.16) \quad w(x_m, t) \leq 0, \quad w(x_n, t) \leq 0$$

for $0 < t < b$ and, by virtue of the weak convergence,

$$\liminf_{t \rightarrow 0+} \int_{x_m}^{x_n} f(y)w(y, t)dy \leq \lim_{t \rightarrow 0+} \int_{x_m}^{x_n} f(y)u(y, t)dy = 0$$

for all non-negative continuous functions $f(x)$. Lemma 8 yields $u \leq v$ in the rectangle bounded by the lines $x = x_m$, $x = x_n$, $t = 0$, $t = b$, and letting $x_m \rightarrow -\infty$, $x_n \rightarrow \infty$ we conclude that $u \leq v$ in the strip $0 < t < b$. Hence $u \in H^\Delta$ in this strip. The weak convergence also implies that

$$\lim_{t \rightarrow 0+} \int_{y_1}^{y_2} u(y, t)dy = 0$$

for all y_1, y_2 and, with Corollary 9, we obtain $u \equiv 0$ in $0 < t < b$. If $b < c$, the proof is completed by a step-by-step argument.

The preceding argument suggests still another one-sided variant of the Tychonoff theorem similar to Theorem 11.

THEOREM 15. *Suppose that $u \in H$ in $0 < t < c$ and that $\{x_n\}$ is as defined in Theorem 14. If (5.15) holds for all n , $0 < t < c$, and if*

$$(5.17) \quad \limsup_{(y,t) \rightarrow (x,0)} u(y,t) \leq A$$

for all x , then $u \leq A$ in $0 < t < c$.

Proof. Set $A = 0$ and let b , v , and w be as defined in the proof of Theorem 14. Next fix $x_m < x_n$; clearly (5.16) holds for $0 < t < b$. Then (5.17) implies that

$$\limsup_{(y,t) \rightarrow (x,0)} w(y,t) \leq 0$$

for all x and, with Lemma 6, we obtain $u \leq v$ in the rectangle bounded by the lines $x = x_m$, $x = x_n$, $t = 0$, $t = b$. Arguing as before we conclude that $u \in H^A$ in $0 < t < b$ and Theorem 10 yields $u \leq A$ in this strip.

COROLLARY 10. *Suppose that $u \in H$ in $0 < t < c$ and that $\{x_n\}$ is as defined in Theorem 14. If (5.15) holds for all n , $0 < t < c$, and if*

$$\lim_{(y,t) \rightarrow (x,0)} u(y,t) = A$$

for all x , then $u \equiv A$ in $0 < t < c$.

6. Fejer-Riesz inequality. We conclude this paper with a heat equation analogue of the following well known inequality [4] and [14].

FEJER-RIESZ INEQUALITY. *Suppose that $u = u(z)$ is harmonic in $|z| < 1$. Then, for each $0 < \rho < 1$,*

$$\int_{-\rho}^{\rho} |u_x(x)| dx \leq \frac{1}{2} \int_{-\pi}^{\pi} |u_{\theta}(\rho e^{i\theta})| d\theta.$$

This inequality has an important interpretation, namely that under any conformal mapping of the unit disk onto a Jordan domain, the length of the image of a diameter never exceeds one-half the length of the image of the unit circle.

The following result suggests a new interpretation for this inequality.

THEOREM 16. *Suppose that $u \in H^A$ in $0 < t < \infty$. Then, for each $0 < c < \infty$,*

$$(6.1) \quad \int_c^{\infty} |u_x(0,t)| dt \leq \frac{1}{2} \int_0^{\infty} |u(x,c) - u(-x,c)| dx \leq \frac{1}{2} \int_{-\infty}^{\infty} |u(x,c)| dx,$$

$$(6.2) \quad \int_c^{\infty} |u_t(0,t)| dt \leq \frac{1}{2} \int_0^{\infty} |u_x(x,c) - u_x(-x,c)| dx \leq \frac{1}{2} \int_{-\infty}^{\infty} |u_x(x,c)| dx.$$

Proof. We begin by considering (6.1) in the special case where, for a fixed y , $u(x, t) = k(x - y, t)$. In particular we observe that, with $s = |y|/2t^{1/2}$,

$$(6.3) \quad \int_0^\infty |k_x(y, t)| dt = \begin{cases} \frac{1}{\pi^{1/2}} \int_0^\infty e^{-s^2} ds = \frac{1}{2} & y \neq 0, \\ 0 & y = 0. \end{cases}$$

For the general case fix $0 < c < \infty$. Then, since $u \in H^\Delta$ in $0 < t < \infty$, we can write

$$u(x, t + c) = \int_{-\infty}^\infty k(x - y, t) u(y, c) dy,$$

where the integral is absolutely convergent in $0 < t < \infty$. Differentiating under the integral sign [19], we obtain

$$(6.4) \quad u_x(x, t + c) = \int_{-\infty}^\infty k_x(x - y, t) u(y, c) dy,$$

and, with (6.3) and the Fubini theorem, we conclude that

$$\begin{aligned} \int_0^\infty |u_x(0, t + c)| dt &= \int_0^\infty \left| \int_{-\infty}^\infty k_x(-y, t) u(y, c) dy \right| dt \\ &\leq \int_0^\infty \left\{ \int_0^\infty |k_x(y, t)| dt \right\} |u(y, c) - u(-y, c)| dy \\ &= \frac{1}{2} \int_0^\infty |u(y, c) - u(-y, c)| dy. \end{aligned}$$

Hence we obtain (6.1). For (6.2) we can differentiate (6.4) with respect to x to obtain

$$(6.5) \quad u_{ix}(x, t + c) = \int_{-\infty}^\infty k_{xx}(x - y, t) u(y, c) dy.$$

If $0 < t_1, t_2 < \infty$, we obtain

$$(6.6) \quad k(x_1, t_1) k(x_2, t_2) \leq \left(\frac{t_1 + t_2}{4\pi t_1 t_2} \right)^{1/2} k(x_1 + x_2, t_1 + t_2)$$

from [13] or direct calculation. Since

$$|u(y, c)| \leq \int_{-\infty}^\infty |k(y - z, c)| d\alpha(z),$$

we conclude from (6.6), with $x_1 = x - y$, $x_2 = y - z$, $t_1 = t$, $t_2 = c$, that

$$(6.7) \quad |u(y, c)| k(x - y, t) \leq \left(\frac{c + t}{4\pi ct} \right)^{1/2} \int_{-\infty}^\infty |k(x - z, t + c)| d\alpha(z) < \infty$$

for each fixed (x, t) in $0 < t < \infty$. With (6.7) it is easy to show that

$$u(y, c)k_x(x - y, t) = o(1)$$

as $|y| \rightarrow \infty$, and integration by parts in (6.5) yields

$$u_t(x, t + c) = \int_{-\infty}^{\infty} k_x(x - y, t)u_y(y, c)dy.$$

The proof for (6.2) is then completed as before.

Both (6.1) and (6.2) have physical interpretations, if we think of u as the temperature, on an absolute scale, of an infinite insulated rod with unit cross-section and unit thermal diffusivity. For (6.1), suppose that at time $t=c$ the total heat in the rod is finite and equal to A , i.e.

$$\int_{-\infty}^{\infty} u(x, c)dx = A.$$

Then it follows that the amount of heat which crosses any fixed section of the rod, in the time interval $c \leq t < \infty$, never exceeds $A/2$. For (6.2), suppose that at time $t=c$ the temperature variation along the rod is bounded and equal to A . Then at each section of the rod the temperature variation in time, for $c \leq t < \infty$, never exceeds $A/2$.

REFERENCES

1. G. Birkhoff and J. Kotik, *Note on the heat equation*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 162-167.
2. J. L. B. Cooper, *The uniqueness of the solution of the equation of heat conduction*, J. London Math. Soc. vol. 25 (1950) pp. 173-180.
3. J. Czipser, *Sur la propagation de la chaleur dans une barre infinie*, I, Magyar Tud. Akad. Alkalm. Mat. Int. Közl. vol. 3 (1954) pp. 395-408 (Hungarian, Russian and French summaries).
4. L. Fejér and F. Riesz, *Über einige funktionentheoretische Ungleichungen*, Math. Z. vol. 11 (1921) pp. 305-314.
5. W. Fulks, *On the unique determination of solutions of the heat equation*, Pacific J. Math. vol. 3 (1953) pp. 387-391.
6. F. W. Gehring, *The Fatou theorem and its converse*, Trans. Amer. Math. Soc. vol. 85 (1957) pp. 106-121. *Errata*, ibid., vol. 86 (1957) p. 532.
7. ———, *On solutions of the equation of heat conduction*, Michigan Math. J. vol. 5 (1958) pp. 191-202.
8. G. H. Hardy, *Divergent series*, Oxford, 1949.
9. P. Hartman and A. Wintner, *On solutions of the equation of heat conduction*, Amer. J. Math. vol. 72 (1950) pp. 367-395.
10. E. L. Ince, *Ordinary differential equations*, New York, 1944.
11. A. J. Lohwater, *A uniqueness theorem for a class of harmonic functions*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 278-279.
12. L. H. Loomis, *The converse of the Fatou theorem for positive harmonic functions*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 239-250.
13. D. Resch, *Temperature bounds on the infinite rod*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 632-634.

14. F. Riesz, *Eine Ungleichung für harmonische Funktionen*, Monatshefte Math. u. Physik vol. 43 (1936) pp. 401–406.
15. P. C. Rosenbloom, *Linear equations of parabolic type with constant coefficients*, Annals of Mathematics Studies, no. 33, Princeton, 1954.
16. S. Saks, *Theory of the integral*, Warsaw, 1937.
17. S. Täcklind, *Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique*, Nova Acta Soc. Sci. Upsal. vol. 10 (1936) pp. 1–57.
18. A. Tychonoff, *Théorèmes d'unicité pour l'équation de la chaleur*, Rec. Math. (Mat. Sb.) vol. 42 (1935) pp. 199–216.
19. D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 85–95.
20. ———, *The Laplace transform*, Princeton, 1946.
21. ———, *Integral transforms related to heat conduction*, Ann. Mat. Pura Appl. vol. 42 (1956) pp. 279–305.

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